Discrete Ordinate Method for Solving Inhomogeneous Vector Radiative Transfer Equation

We describe here a solution method for equations of the type given by:

$$\mu \frac{\partial \mathbf{I}(\tau,\mu)}{\partial \tau} + \mathbf{I}(\tau,\mu) - \frac{\omega(\tau)}{2}\int_{-1}^{1} \mathbf{Z}(\tau,\mu,\mu')\mathbf{I}(\tau,\mu')d\mu' = \mathbf{Q}(\tau,\mu),$$

where $\mathbf{I}$ and $\mathbf{Q}$ are Stokes vectors, and the $\mathbf{Z}$ are Mueller matrices as detailed in the main paper.. This type of equation appears when modeling radiative transport in plane parallel media. We will assume here that the plane parallel medium is described by one or more homogeneous layers (see Figure 1), that means the function $\mathbf{Z}$ and ω in each layer are constant in one layer. For simplicity of explanation, we first describe solution methods for the equation corresponding to a single homogeneous layer. We then generalize the methods to account for multiple layers. The above equation for a single homogeneous layer of thickness τ_0 can be rewritten as:

$$\mu \frac{\partial \mathbf{I}(\tau,\mu)}{\partial \tau} + \mathbf{I}(\tau,\mu) - \frac{\omega}{2}\int_{-1}^{1} \mathbf{Z}(\mu,\mu')\mathbf{I}(\tau,\mu')d\mu' = \mathbf{Q}(\tau,\mu) \quad (1)$$

or in a compact operator notation as:

$$\mathscr{L}[\mathbf{I}(\tau,\mu)] = \mathbf{Q}(\tau,\mu) \quad (2)$$

where

$$\mathbf{Q}(\tau,\mu) = \frac{\omega}{2}\mathbf{Z}(\tau,\mu,\mu_{inc})\mathbf{I}_{inc}(\mu_{inc})e^{-\tau/\mu_{inc}} \quad (3)$$

and $\mathscr{L}$ an integro differential operator as expanded in equation 1.

In classic differential equations literature, equations similar to (2) are called inhomogeneous equations, and $\mathbf{Q}$'s on the right hand side are called the inhomogeneous terms. A standard way of computing the solution to inhomogeneous equations is: find solution $\mathbf{I}_h$ to the homogeneous form of the equation such that,

$$\mathscr{L}[\mathbf{I}_h(\tau,\mu)] = 0 \quad (4)$$

and then find a particular solution $\mathbf{I}_p$ that satisfies the original equation, i.e.

$$\mathscr{L}[\mathbf{I}_p(\tau,\mu)] = \mathbf{Q}(\tau,\mu) \quad (5)$$

and compose the final solution as a combination of the two, i.e. $\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p$ (definition 1.15 in [1], page 6).

Applying Gaussian quadrature scheme to the integral term in equation 1 gives us a system of N equations

$$\mu_i \frac{\partial \mathbf{I}(\tau,\mu_i)}{\partial \tau} + \mathbf{I}(\tau,\mu_i) - \frac{\omega}{2}\sum_{j=1}^{N} \alpha_j \mathbf{Z}(\mu_i,\mu_j)\mathbf{I}(\tau,\mu_j) = \mathbf{Q}(\tau,\mu_i) \quad (6)$$

where μ_i's and α_i's are the Gaussian quadrature angles and weights respectively. So we must solve this system of equations.

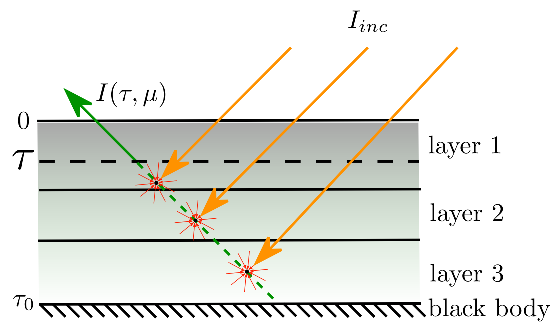

Figure 1: Material composed of three plane-parallel layers. The computation of the radiance at an optical depth of τ in a direction μ requires solving light transport in each layer.It involves the computation of the scattered radiance along μ from all direction (in red) and the scattering of the attenuated incident radiance (in yellow).

1 Computing homogeneous solution I_h

The first step in computing $\mathbf{I}(\tau,\mu_i)$, the solutions of equation 6, is to compute a homogeneous solution $\mathbf{I}_h$ that satisfies:

$$\mu_i \frac{\partial \mathbf{I}_h(\tau,\mu_i)}{\partial \tau} + \mathbf{I}_h(\tau,\mu_i) - \frac{\omega}{2}\sum_{j=1}^{N} \alpha_j \mathbf{Z}(\mu_i,\mu_j)\mathbf{I}_h(\tau,\mu_j) = 0 \quad (7)$$

Such an equation is known to have exponential solutions, so we rewrite $\mathbf{I}$ as [2]:

$$\mathbf{I}_h(\tau,\mu_i) = \Phi(\mu_i)e^{-\tau/\nu}$$

where $\Phi(\mu_i)$'s are 4 element vectors and ν are scalars, both being unknowns.

Substituting this in equation 7 and using the relation

$$\frac{\partial \mathbf{I}_h(\tau,\mu_i)}{\partial \tau} = -\frac{1}{\nu}\Phi(\mu_i)e^{-\tau/\nu} \quad (8)$$

we get a linear system of equations:

$$-\frac{\mu_i}{\nu}\Phi(\mu_i) + \Phi(\mu_i) - \frac{\omega}{2}\sum_{j=1}^{N} \alpha_j \mathbf{Z}(\mu_i,\mu_j)\Phi(\mu_j) = 0, \quad (9)$$

When written in a matrix form, (9) gives us an eigenproblem:

$$\mathscr{M}\begin{bmatrix}\Phi(\mu_1)\\ \Phi(\mu_2)\\ \vdots \\ \Phi(\mu_N)\end{bmatrix} = \frac{1}{\nu}\begin{bmatrix}\Phi(\mu_1)\\ \Phi(\mu_2)\\ \vdots \\ \Phi(\mu_N)\end{bmatrix} \quad (10)$$

where each 4×4 subblock of $\mathscr{M}$ is defined as:

$$\mathscr{M}_{ij} = \frac{1}{\mu_i}\left(\mathbf{I}_4 - \frac{\omega}{2}\alpha_j \mathbf{Z}(\mu_i,\mu_j)\right) \quad (11)$$

$\mathbf{I}_4$ being a 4×4 identity matrix.

Solution of this eigenproblem gives us a set of $4N$ eigenvectors Φ_i and eigenvalues ν_i. So we express our solution to equation 7 as a linear combination of the eigenvectors:

$$\mathbf{I}_h(\tau,\mu_i) = \sum_{k=1}^{4N} L_k \Phi_k(\mu_i) e^{-\tau/\nu_i}. \tag{12}$$

where L_k's are the scalar factors that must be determined.

Thus the computation of the homogeneous solution requires eigensolution of a matrix of size $4N \times 4N$.

2 Computing particular solution

One common approach to finding a particular solution I_p to the inhomogeneous equation 6 is to express it in the same form as the inhomogeneous source term $Q(\tau,\mu_i)$. To simplify the discussion we rewrite $Q(\tau,\mu_i)$ shown in (3) as:

$$\mathbf{Q}(\tau,\mu_i) = \mathbf{X}(\mu_i) e^{-\tau/\mu_{inc}} \tag{13}$$

So the particular solution we seek can be expressed as:

$$\mathbf{I}_p(\tau,\mu_i) = \mathbf{Y}(\mu_i) e^{-\tau/\mu_{inc}} \tag{14}$$

where $\mathbf{Y}(\mu_i)$'s vectors of 4 unknown elements.

Substituting (13) and (14) in (6) and using the relationship

$$\frac{\partial \mathbf{I}_p(\tau,\mu_i)}{\partial \tau} = -\frac{1}{\mu_{inc}} \mathbf{Y}(\mu_i) e^{-\tau/\mu_{inc}} \tag{15}$$

we get:

$$(1 - \frac{\mu_i}{\mu_{inc}})\mathbf{Y}(\mu_i) - \sum_{j=1}^{N} \alpha_j \mathbf{Z}(\mu_i,\mu_j)\mathbf{Y}(\mu_j) = \mathbf{X}(\mu_i) \tag{16}$$

So we have a system of $4N$ linear equations with N unknown vectors $\mathbf{Y}$, so a total of $4N$ unknowns. We can use a linear system solver to compute these unknowns.

3 Computing I

Having computed the homogeneous solution and the particular solution, we can compose our I terms as

$$\mathbf{I}(\tau,\mu_i) = \mathbf{I}_h(\tau,\mu_i) + \mathbf{I}_p(\tau,\mu_i) \tag{17}$$

Note that we still have $4N$ unknown L_k's in the expression of $\mathbf{I}_h$ (see equation 12). We finally compute these unknowns by using the boundary conditions at $\tau = 0$ and at $\tau = \tau_0$.

Assuming that the incident radiance field at the top of the material volume layer is zero, and that the material is placed on top of a black body so that no light is entering from the bottom, the boundary conditions are as follow.

$$\mathbf{I}_h(0,\mu_i) + \mathbf{I}_p(0,\mu_i) = 0 \tag{18}$$

where μ_i's are negative, and

$$\mathbf{I}_h(\tau_0,\mu_i) + \mathbf{I}_p(\tau_0,\mu_i) = 0, \tag{19}$$

where μ_i's are positive.

Each of these boundary conditions account for $N/2$ equations involving 4 elements vectors, so a total of $4N$ equations for $4N$ unknowns. We solve for the unknown L_k's by solving the system of equations.

Note that it is not required to assume a black body interface at the bottom of the layer. If we know the BRDF of the base material on top of which our plane parallel material is placed, then we can compute the radiance field for the positive angles at $\tau = \tau_0$ from the the radiance field for the negative angles at the same location and the BRDF of the base material, and get the required $N/2$ equations.

4 Multiple layers

At this point we know how to solve equation 1 for a single homogeneous layer. These computations can be extended to a material composed of N_z homogeneous layers placed one on top of the other. The single layer computation discussed earlier can be applied independently to find $\mathbf{I}$'s at each layer as a combination of both homogeneous and particular solutions for the layer. However, one thing remains to be computed: the N unknown constant L_k's for each layer that are used to combine individual eigenvector based homogenous solutions (see equation 12). In section 3, we discussed how to compute them for a single layer. To extend that method to multiple layers, we need to compute $N_z \times N$ unknowns and so we need $N_z \times N$ linear equations to solve for these unknowns. The two boundary conditions (18) and (19) of the top surface and bottom surface of the first and the last layer respectively make N linear equations. The remaining $(N_z - 1) \times N$ equations come from the $(N_z - 1)$ interfaces between N_z layers. The function I must be continuous at the interface between the layers. Therefore, using $\mathbf{I}_z(\tau)$ to denote the $\mathbf{I}$ field at layer z, and τ_z to denote the thickness at the bottom of that same layer we write:

$$\mathbf{I}_z(\tau_z,\mu_i) = \mathbf{I}_{z+1}(\tau_z,\mu_i), \quad z \in \{1,\cdots,N_z-1\}, i \in \{1,\cdots,N\} \tag{20}$$

Thus in total we get $N_z \times N$ equations. We solve this linear set of equations to compute the unknown L_k's for all the N_z layers.

5 Algorithm

We summarize all the steps of the solution in algorithm 1.

Algorithm 1 main()

```
1: ComputeHomogeneous();
2: ComputeParticular(μ_inc);
3: ComputeRadianceField(μ_inc);
```

`ComputeHomogeneous()` solves the eigenproblem defined in section 1, giving us all the eigenvectors (Φ's) and eigenvalues (ν's). `ComputeParticular()` computes the unknown vector $\mathbf{Y}$ (14) for every μ_{inc} of the incident direction, so that the particular solution can be fully reconstructed. `ComputeRadianceField()` computes the constants L_k using the method presented in section 3. The homogeneous solution is independent of the incident direction, therefore when applying this algorithm to BRDF computations, one would solve the homogeneous problem only once and re-use the same solution for each incident direction.

References

[1] C. Constanda. *Differential Equations, A Primer for Scientists and Engineer*. Springer Undergraduate Texts in Mathematics and Technology, first edition, 2013.

[2] K. Stamnes and R. A. Swanson. A new look at the discrete ordinate method for radiative transfer calculations in anisotropically scattering atmospheres. *Journal of the Atmospheric Sciences*, 38(2):387–399, 1981.